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ORIGINAL RESEARCH

On a p-Kirchhoff-type problem arising in ecosystems

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Abstract In this article, we discuss the existence of positive solutions for an ecological model of the form:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p \,dx\right)\Delta_p u = \dfrac{au^{p-1} - bu^{\gamma-1} - c}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \gamma$, $M : [0, \infty) \longrightarrow (0, \infty)$ is a continuous and increasing function, $a > 0$, $b > 0$, $c \geq 0$, and $\alpha \in (0, 1)$. This model describes the steady states of a logistic growth model with grazing and constant yield harvesting. It also describes the dynamics of the fish population with natural predation and constant yield harvesting. We discuss the existence of a positive solution for given a, b, γ and small values of c.

Keywords Positive solutions · Sub-supersolutions · p-Kirchhoff-type problems

Mathematics Subject Classification 35J55 · 35J65

Introduction

In this paper, we are interested in the existence of positive solutions for the p-Kirchhoff-type problems

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p \,dx\right)\Delta_p u = \dfrac{au^{p-1} - bu^{\gamma-1} - c}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $M : [0, \infty) \longrightarrow (0, \infty)$ is a continuous and increasing function, $c \geq 0$, $a, b > 0$, Ω is a bounded domain with smooth boundary, Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = \mathrm{div}\,(|\nabla z|^{p-2}\nabla z)$, $1 < p < \gamma$ and $\alpha \in (0, 1)$.

Here u is the population density and $\frac{au^{p-1} - bu^{\gamma-1}}{u^\alpha}$ represents logistics growth. This model describes grazing of a fixed number of grazers on a logistically growing species (see [11]). The herbivore density is assumed to be a constant which is a valid assumption for managed grazing systems and the rate of grazing is given by $\frac{c}{u^\alpha}$. At high levels of vegetation density this term saturates to c as the grazing population is a constant. This model has also been applied to describe the dynamics of fish populations (see [15]). In the case of the fish population the term $\frac{c}{u^\alpha}$ corresponds to natural predation.

In recent years, problems involving Kirchhoff-type operators have been studied in many papers, we refer to [3, 4, 6, 10, 14] in which the authors have used the variational and topological methods to get the existence of solutions. In this article, we are motivated by the ideas introduced in [7, 12, 13] and properties of Kirchhoff-type operators in [3, 4, 6], we study problem (1) in semipositone case (i.e., $\lim_{s\longrightarrow 0^+} f(s) = -\infty$; $f(s) = \frac{as^{p-1} - bs^{\gamma-1} - c}{s^\alpha}$); see [5, 7–9]). Using sub-supersolution techniques, we prove the existence of a positive solution for the problem.

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta_p \phi = \lambda \,|\, \phi \,|^{p-2}\, \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (3) such that $\phi(x) > 0$ in Ω and $||\phi||_\infty = 1$. It can be shown that $\frac{\partial \phi}{\partial n} < 0$ on $\partial\Omega$. Here n is the outward normal. Let $m, \delta > 0$ and $\mu > 0$ be such that:

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$$\mu \leq \phi \leq 1, \quad x \in \Omega - \overline{\Omega_\delta}, \tag{3}$$

$$|\nabla \phi|^p \geq m, \quad x \in \overline{\Omega_\delta}, \tag{4}$$

with $\overline{\Omega_\delta} := \{x \in \Omega | d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla\phi|^p \neq 0$ on $\partial\Omega$ while $\phi = 0$ on $\partial\Omega$. We will also consider the unique solution $e \in W_0^{1,p}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_p e = 1, & x \in \Omega, \\ e = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result, it is known that $e > 0$ in Ω and $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

Existence results

In this section, we shall establish our existence result via the method of sub-supersolution. A function ψ is said to be a subsolution of (1), if it is in $W_0^{1,p}(\Omega)$ such that

$$- M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \int_\Omega |\nabla\psi|^{p-2} \nabla\psi \cdot \nabla w dx \leq \int_\Omega \left[\frac{a\psi^{p-1} - b\psi^{\gamma-1} - c}{\psi^\alpha}\right] w dx,$$

and z is said supersolution of (1), if it is in $W_0^{1,p}(\Omega)$ such that

$$- M\left(\int_\Omega | \nabla z |^p \, dx\right) \int_\Omega |\nabla z|^{p-2} \nabla z \cdot \nabla w dx \geq \int_\Omega \left[\frac{az^{p-1} - bz^{\gamma-1} - c}{z^\alpha}\right] w dx,$$

for all $w \in W = \{w \in C_0^\infty(\Omega) | w \geq 0, x \in \Omega\}$. Then the following result holds:

Then the following result holds:

Lemma 2.1 (See [1, 2, 8]) Suppose there exist sub and supersolutions ψ and z respectively of (1) such that $\psi \leq z$. Then (1) has a solution u such that $\psi \leq u \leq z$

Now we state our main result.

Theorem 2.2 Let there exist constants $M_0 > 0$ and $M_\infty \geq 0$ such that $M_0 \leq M(t) \leq M_\infty$ for all $t \in [0, \infty)$. Given $a, b > 0$, $1 < p < \gamma$, and $\alpha \in (0, 1)$, there exists a constant $c_1 = c_1(a, b, \alpha, \gamma, p, \Omega) > 0$ such that for $c < c_1$, (1) has a positive solution.

Remark 2.3 In the nonsingular case ($\alpha = 0$), positive solutions exist only when $a > \lambda_1$(the principle eigenvalue) (see [12, 13]). But in the singular case, we establish the existence of a positive solution for any $a > 0$.

Proof of Theorem 2.2 We start with the construction of a positive subsolution for (1). Fix $\beta \in (1, \frac{p}{p-1+\alpha})$. Define $\psi = k\phi^\beta$, where $k > 0$ is such that $a \geq 2bk^{\gamma-p} + M_\infty \beta^{p-1} \lambda_1 k^\alpha$. Define

$$c_1 := \min\left\{M_\infty k^{p-1+\alpha} \beta^{p-1} (\beta - 1)(p-1) m^p, \right.$$
$$\left. \frac{1}{2} M_\infty k^{p-1} \mu^{\beta(p-1)} \left(a - \beta^{p-1} \lambda_1 k^\alpha\right)\right\}.$$

Note that $c_1 > 0$ by the choice of k and β. A calculation shows that

$$\nabla\psi = k\beta\phi^{\beta-1},$$

and

$$\begin{aligned}
&-M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \Delta_p \psi \\
&= M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \int_\Omega |\nabla\psi|^{p-2} \nabla\psi \cdot \nabla w dx \\
&= k^{p-1} \beta^{p-1} M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \\
&\quad \times \int_\Omega \phi^{(\beta-1)(p-1)} |\nabla\phi|^{p-2} \nabla\phi \nabla w dx \\
&= k^{p-1} \beta^{p-1} M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \\
&\quad \times \int_\Omega |\nabla\phi|^{p-2} \nabla\phi \left[\nabla(\phi^{(\beta-1)(p-1)} w) - w \nabla(\phi^{(\beta-1)(p-1)}\right] dx \\
&= k^{p-1} \beta^{p-1} M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \\
&\quad \times \int_\Omega |\nabla\phi|^{p-2} \nabla\phi \nabla(\phi^{(\beta-1)(p-1)} w) dx \\
&\quad - k^{p-1} \beta^{p-1} M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \\
&\quad \times \int_\Omega |\nabla\phi|^{p-2} \nabla\phi \nabla(\phi^{(\beta-1)(p-1)}) w dx \\
&= k^{p-1} \beta^{p-1} M\left(\int_\Omega | \nabla\psi |^p \, dx\right) \\
&\quad \times \int_\Omega \left[\lambda_1 \phi^{\beta(p-1)} - (\beta-1)(p-1) |\nabla\phi|^p \phi^{-p+\beta(p-1)}\right] w dx \\
&\leq M_\infty k^{p-1} \beta^{p-1} \int_\Omega [\lambda_1 \phi^{\beta(p-1)} \\
&\quad - (\beta-1)(p-1) |\nabla\phi|^p \phi^{-p+\beta(p-1)}] w dx.
\end{aligned}$$

Thus ψ is a subsolution of (1) if

$$M_\infty k^{p-1} \beta^{p-1} \left[\lambda_1 \phi^{\beta(p-1)} - (\beta-1)(p-1) |\nabla\phi|^p \phi^{-p+\beta(p-1)}\right]$$
$$\leq ak^{p-1-\alpha} \phi^{\beta(p-1-\alpha)} - bk^{\gamma-1-\alpha} \phi^{\beta(\gamma-1-\alpha)} - \frac{c}{k^\alpha \phi^{\alpha\beta}}.$$

For this, we have to show the following three inequalities:

$$- k^{p-1-\alpha}\phi^{\beta(p-1-\alpha)}\left(a - M_\infty k^\alpha \beta^{p-1}\lambda_1\phi^{\alpha\beta}\right) \le -2bk^{\gamma-1-\alpha}\phi^{\beta(\gamma-1-\alpha)}, \quad x\in \ \Omega,$$

$$-\frac{1}{2}k^{p-1-\alpha}\phi^{\beta(p-1-\alpha)}(a - M_\infty k^\alpha\beta^{p-1}\lambda_1\phi^{\alpha\beta}) \le -\frac{c}{k^\alpha\phi^{\alpha\beta}}, \quad x\in \ \Omega - \overline{\Omega_\delta},$$

$$- M_\infty k^{p-1}\beta^{p-1}(p-1)(\beta-1)\frac{|\nabla\phi|^p}{\phi^{p-\beta(p-1)}} \le -\frac{c}{k^\alpha\phi^{\alpha\beta}}, \quad x\in\overline{\Omega_\delta},$$

by the choice of k, we have:

$$-\frac{1}{2}k^{p-1-\alpha}\phi^{\beta(p-1-\alpha)}(a - M_\infty k^\alpha\beta^{p-1}\lambda_1\phi^{\alpha\beta}) \le -bk^{\gamma-1-\alpha}\phi^{\beta(p-1-\alpha)} \le -bk^{\gamma-1-\alpha}\phi^{\beta(\gamma-1-\alpha)}. \tag{5}$$

Now, we have in $\overline{\Omega_\delta}$, $|\nabla\phi|^p \ge m$, and $c < M_\infty k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p$, then the following inequalities hold:

$$- M_\infty k^{p-1}\beta^{p-1}(p-1)(\beta-1)\frac{|\nabla\phi|^p}{\phi^{p-\beta(p-1)}} \le \frac{-M_\infty k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{k^\alpha\phi^{\alpha\beta}\phi^{p-\beta(p-1)-\alpha\beta}} \le -\frac{c}{k^\alpha\phi^{\alpha\beta}\phi^{p-\beta(p-1)-\alpha\beta}}.$$

On the other hand, since $p-\beta(p-1+\alpha) > 0$,

$$-\frac{c}{k^\alpha\phi^{\alpha\beta}\phi^{p-\beta(p-1)-\alpha\beta}} \le -\frac{c}{k^\alpha\phi^{\alpha\beta}}.$$

Hence

$$-M_\infty k^{p-1}\beta^{p-1}(p-1)(\beta-1)\frac{|\nabla\phi|^p}{\phi^{p-\beta(p-1)}} \le -\frac{c}{k^\alpha\phi^{\alpha\beta}}. \tag{6}$$

Finally, in $\Omega - \overline{\Omega_\delta}$ using $\phi \ge \mu$ and $c < \frac{1}{2}M_\infty k^{p-1}\mu^{\beta(p-1)}(a - \beta^{p-1}\lambda_1 k^\alpha)$, we have:

$$-\frac{1}{2}k^{p-1-\alpha}\phi^{\beta(p-1-\alpha)}(a - M_\infty k^\alpha\beta^{p-1}\lambda_1\phi^{\alpha\beta}) \le \frac{-k^{p-1}\phi^{\beta(p-1)}(a - M_\infty k^\alpha\beta^{p-1}\lambda_1)}{2k^\alpha\phi^{\alpha\beta}} \le -\frac{c}{k^\alpha\phi^{\alpha\beta}}. \tag{7}$$

For $c < c_1$, by (6) and (7) the Eq. (5) holds. Thus ψ is a subsolution of (1).

Now for a supersolution choose $z := Ne$, where $N > 0$ is such that $Ne \ge \psi$ and

$$\frac{au^{p-1} - bu^{\gamma-1} - c}{M_0 u^\alpha} \le N^{p-1},$$

for all $u > 0$. We have

$$-M\left(\int_\Omega |\nabla z|^p \, \mathrm{d}x\right)\triangle_p z = M\left(\int_\Omega |\nabla z|^p \, \mathrm{d}x\right)N^{p-1} \ge M_0 N^{p-1} \ge \left[\frac{az^{p-1} - bz^{\gamma-1} - c}{z^\alpha}\right].$$

i.e., z is a supersolution of (1) with $z \ge \psi$ for N large (note $|\nabla e| \ne 0; \partial\Omega$). Thus, there exists a positive solution u of (1) such that $\psi \le u \le z$. This completes the proof of Theorem 2.2. □

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